

CHERN SUBRINGS

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ABSTRACT. Let p be an odd prime. We show that for a simply-connected semisimple complex linear algebraic group, if its integral homology has p -torsion, the Chern classes do not generate the Chow ring of its classifying space.

1. INTRODUCTION

Let p be an odd prime. Let $h^*(-)$ be one of the mod p cohomology $H\mathbb{Z}/p$, the cohomology $H\mathbb{Z}_{(p)}$ with coefficient $\mathbb{Z}_{(p)}$ and the Brown-Peterson cohomology BP with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$. Let G be a compact connected Lie group and $G(\mathbb{C})$ its complexification, that is, $G(\mathbb{C})$ is a complex linear algebraic group which is homotopy equivalent to the compact connected Lie group G . Considering a finite dimensional complex representation $\rho : G \rightarrow GL_m(\mathbb{C})$, we have Chern classes $c_i(\rho)$ in the cohomology $h^*(BG)$ of classifying space and the Chern subring $Ch_h(G) \subset h^*(BG)$, a subalgebra over h_* generated by Chern classes, where ρ ranges over all finite dimensional representations. If G is one of classical groups $SU(n)$, $Spin(n)$ and $Sp(n)$, the cohomology $h^*(BG)$ is generated by Chern classes and $h^*(BG) = Ch_h(G)$ for arbitrary odd prime p .

The case of the Brown-Peterson cohomology is particularly interesting in conjunction with the study of Chow rings of classifying spaces of complex linear algebraic groups defined by Totaro. In [To], Totaro considered the classifying space of the linear algebraic group $G(\mathbb{C})$ as a limit of algebraic varieties, defined the Chow ring for it and showed that the cycle map factors through the Brown-Peterson cohomology,

$$CH^*(BG(\mathbb{C}))_{(p)} \rightarrow BP^*(BG) \otimes_{BP_*} \mathbb{Z}_{(p)} \rightarrow H^{even}(BG; \mathbb{Z}_{(p)}),$$

where $H^{even}(BG; \mathbb{Z}_{(p)})$ is the direct sum of $H^{2i}(BG; \mathbb{Z}_{(p)})$ ($i \geq 0$). He also conjectured that the left homomorphism $CH^*(BG(\mathbb{C}))_{(p)} \rightarrow BP^*(BG) \otimes_{BP_*} \mathbb{Z}_{(p)}$ is an isomorphism. We may consider a Chern subring for the Chow ring $CH^*(BG(\mathbb{C}))$ as in the case of the above $Ch_h(G)$.

In [Ka-Ya] and [Vi], the Chow ring $CH^*(BPGL_p(\mathbb{C}))_{(p)}$ of the complex linear algebraic group $PGL_p(\mathbb{C})$, which is the complexification of the projective unitary group $PU(p)$, and related cohomology theories were computed and it was shown that

$$CH^*(BPGL_p(\mathbb{C}))_{(p)} = BP^*(BPU(p)) \otimes_{BP_*} \mathbb{Z}_{(p)} = H^{even}(BPU(p); \mathbb{Z}_{(p)})$$

through the cycle map above. In [Ka-Ya, Proposition 5.7], we showed similar results for $(p, G) = (3, F_4)$, $(5, E_8)$. For $p = 3$, the computation of the Brown-Peterson cohomology was done by Kono and Yagita in [Ko-Ya] and Kono and Yagita showed that x_8^a is not in the Chern subring unless a is divisible by 2. In [Ta], Targa showed

that x_{2p+2}^a in $CH^*(BPGL_p(\mathbb{C}))_{(p)}$, where $a \leq p-2$, is not in the Chern subring for arbitrary odd prime p .

In this paper, we prove the following and generalize the above computation of Kono, Yagita and Targa. Let Q_i be the Milnor operations of degree $2p^i - 1$ which acts on the mod p cohomology of a space.

Theorem 1.1. *For $(p, G) = (p, PU(p))$, let $x = Q_0 Q_1 x_2$ where x_2 is the generator of $H^2(BG; \mathbb{Z}/p) = \mathbb{Z}/p$. For $(p, G) = (3, F_4), (3, E_6), (3, E_7), (3, E_8), (5, E_8)$, let $x = Q_1 Q_2 x_4$ where x_4 is the generator of $H^4(BG; \mathbb{Z}/p) = \mathbb{Z}/p$. Then, x^a is not in the Chern subring $Ch_{H\mathbb{Z}/p}(G)$ unless a is divisible by $p-1$.*

This theorem implies that if x comes from the Chow ring through the cycle map, then the Chow ring is not generated by Chern classes. Recall that motivic cohomology $H^{*,*'}(BG(\mathbb{C}), \mathbb{Z}/p)$ contains $CH^*(BG(\mathbb{C}))/p$ as

$$CH^*(BG(\mathbb{C}))/p = H^{2*,*}(BG(\mathbb{C}), \mathbb{Z}/p).$$

Moreover, the motivic cohomology has the action of Milnor operations Q_i where the degree of Q_i is $(2p^i - 1, p^i - 1)$. If there exists an element $x_{4,3}$ in $H^{4,3}(BG(\mathbb{C}), \mathbb{Z}/p)$ corresponding to x_4 in $H^4(BG; \mathbb{Z}/p)$, then $x = Q_1 Q_2(x_{4,3})$ is in the Chow ring

$$CH^{p^2+p+1}(BG(\mathbb{C}))/p = H^{2p^2+2p+2, p^2+p+1}(BG(\mathbb{C}), \mathbb{Z}/p)$$

and through the cycle map it maps to x in Theorem 1.1. In [Ya], Lemma 9.6, Yagita proved that if $px_4 \in H^4(BG; \mathbb{Z}_{(p)})$ is a Chern class of some representation, then the element $x_{4,3}$ above exists. In [Sc-Ya], Schuster and Yagita showed that for $(p, G) = (3, F_4)$, $3x_4$ is the Chern class of the complexification of the irreducible representation of F_4 . In this paper, by computing the Chern class of the adjoint representation of E_8 , we prove the following proposition.

Proposition 1.2. *For $(p, G) = (3, F_4), (3, E_6), (3, E_7), (3, E_8)$ and $(5, E_8)$, there exists a complex representation α of G and $\gamma \in \mathbb{Z}_{(p)}^\times$ such that the element $\gamma px_4 \in H^4(BG; \mathbb{Z}_{(p)})$ is a Chern class $c_2(\alpha)$.*

Thus, we have the following result on Chern subrings of Chow rings.

Theorem 1.3. *For $(p, G) = (p, PU(p)), (3, F_4), (3, E_6), (3, E_7), (3, E_8)$ and $(5, E_8)$, the Chow ring $CH^*(BG(\mathbb{C}))_{(p)}$ is not generated by Chern classes.*

In §2, we consider Chern classes of elementary abelian p -groups. In §3, we prove Theorem 1.1. In §4, we prove Proposition 1.2. We thank François-Xavier Dehon for informing us of the work of Targa.

2. CHERN CLASSES OF ELEMENTARY ABELIAN p -GROUPS

In this section, we investigate the total Chern class of finite dimensional complex representation $\rho : A_n \rightarrow GL_m(\mathbb{C})$ of elementary abelian p -group A_n of rank n .

Firstly, we recall the cohomology of BA_n . The mod p cohomology of elementary abelian p -group is a polynomial tensor exterior algebra

$$\mathbb{Z}/p[t_1, \dots, t_n] \otimes \Lambda(dt_1, \dots, dt_n).$$

The elements $dt_1, \dots, dt_n \in H^1(BA_n; \mathbb{Z}/p)$ correspond to the dual of the basis of $\pi_1(BA_n) = H_1(BA_n; \mathbb{Z}/p)$. The elements t_1, \dots, t_n are obtained from dt_1, \dots, dt_n by applying the Milnor operation Q_0 . For the mod p cohomology of a space, there exists an action of Milnor operations Q_0, Q_1, Q_2, \dots and reduced power operations

$\wp^0 = 1, \wp^1, \wp^2, \dots$. The action of Milnor operations on the mod p cohomology of elementary abelian p -group is given by

$$Q_i(dt_k) = t_k^{p^i}, \quad Q_i t_k = 0, \quad Q_i(x \cdot y) = Q_i(x) \cdot y + (-1)^{\deg x} x \cdot Q_i(y).$$

The action of reduced power operations is given by

$$\wp^i dt_k = 0, \quad \wp^i t_k = \begin{cases} t_k^p & (i = 1) \\ 0 & (i \geq 2), \end{cases} \quad \wp^j(x \cdot y) = \sum_{i=0}^j \wp^{i-j} x \cdot \wp^j y.$$

Secondly, we recall the invariant theory of finite general linear groups and special linear groups. The action of Milnor operations commutes with the action of general linear group $GL_n(\mathbb{Z}/p)$ since the action of the general linear group on the mod p cohomology comes from the one on the elementary abelian p -group A_n . For the sake of notational simplicity, we write V_n for the subspace spanned by t_1, \dots, t_n ,

$$V_n = \mathbb{Z}/p\{t_1, \dots, t_n\}.$$

We write SM_n, M_n for Mù invariant

$$H^*(BA_n; \mathbb{Z}/p)^{SL_n(\mathbb{Z}/p)}, \quad H^*(BA_n; \mathbb{Z}/p)^{GL_n(\mathbb{Z}/p)},$$

respectively. We also write SD_n, D_n for Dickson invariants

$$\mathbb{Z}/p[t_1, \dots, t_n]^{SL_n(\mathbb{Z}/p)}, \quad \mathbb{Z}/p[t_1, \dots, t_n]^{GL_n(\mathbb{Z}/p)},$$

respectively. Kameko and Mimura [Ka-Mi] gave a simpler description for SM_n, M_n using Milnor operations. For Dickson invariants and Mù invariant, we refer the reader to [Ka-Mi] and its references. Let us define $c_{n,i}$ for $n = 1, \dots, n-1$ as follows: Consider the polynomial

$$f_n(X) = \prod_{v \in V_n} (X + v)$$

in $\mathbb{Z}/p[t_1, \dots, t_n][X]$. We define $(-1)^{n-i} c_{n,i}$ to be the coefficient of $X^{p^{n-i}}$ in $f_n(X)$. We define e_n by $e_n = Q_0 \cdots Q_{n-1}(dt_1 \cdots dt_n)$. Then, we have the following. For a ring R and for a finite set $\{a_1, \dots, a_r\}$, we denote by $R\{a_1, \dots, a_r\}$ a free R -module with the basis $\{a_1, \dots, a_r\}$.

Proposition 2.1. *There hold the following:*

- (1) $c_{n,0} = e_n^{p-1}$.
- (2) $f_n(X) = X^{p^n} - c_{n,n-1}X^{p^{n-1}} + \cdots + (-1)^n c_{n,0}X$,
- (3) SD_n is a polynomial algebra $\mathbb{Z}/p[e_n, c_{n,n-1}, \dots, c_{n,1}]$.
- (4) D_n is also a polynomial algebra $\mathbb{Z}/p[c_{n,n-1}, \dots, c_{n,1}, c_{n,0}]$.
- (5) M_n is a free D_n -module

$$D_n\{1, e_n^{p-2} dt_1 \cdots dt_n, e_n^{p-2} Q_{i_1} \cdots Q_{i_r}(dt_1 \cdots dt_n)\} \quad \text{and}$$

- (6) SM_n is a free SD_n -module

$$SD_n\{1, dt_1 \cdots dt_n, Q_{i_1} \cdots Q_{i_r}(dt_1 \cdots dt_n)\},$$

where $0 \leq i_1 < \cdots < i_r \leq n-1, 1 \leq r \leq n-1$.

Thirdly, we consider Chern classes. It is well-known that any finite dimensional complex representation of an abelian group is a direct sum of 1-dimensional complex representations. Therefore, the total Chern class $c(\rho)$ is a product of $c(\lambda)$'s where $c(\lambda) = 1 + v$, $v \in V_n$. Thus, the Chern classes are in $\mathbb{Z}/p[t_1, \dots, t_n]$ instead of $H^*(BA_n; \mathbb{Z}/p)$. Let us consider the total Chern class $c(\text{reg})$ of the regular

representation $reg : A_n \rightarrow GL_{p^n}(\mathbb{C})$. It is clear that $GL_n(\mathbb{Z}/p)$ acts on A_n and $c(reg) \in M_n$.

Proposition 2.2. *There holds*

$$c(reg) = \prod_{v \in V_n \setminus \{0\}} (1 + v) = 1 - c_{n,n-1} + \cdots + (-1)^n c_{n,0} \in D_n.$$

For a group W acting $V_n \setminus \{0\}$, we say the action of W is transitive on $V_n \setminus \{0\}$ if and only if for each u, v in $V_n \setminus \{0\}$, there exists $w \in W$ such that $wu = v$. We investigate the total Chern class $c(\rho)$ when the image of the induced homomorphism $B\rho^* : H^*(BGL_m(\mathbb{C}); \mathbb{Z}/0) \rightarrow \mathbb{Z}/p[t_1, \dots, t_n]$ is invariant under certain group action.

Lemma 2.3. *Let $\rho : A_n \rightarrow GL_m(\mathbb{C})$ be a complex representation of elementary abelian p -group A_n of rank n . Suppose that a subgroup W of $GL_n(\mathbb{Z}/p)$ acts on A_n in the obvious manner. Suppose that the total Chern class $c(\rho)$ is in $\mathbb{Z}/p[t_1, \dots, t_n]^W$ and suppose that the action of W on $V_n \setminus \{0\}$ is transitive. Then, $c(\rho) = c(reg)^a$ for some $a \geq 0$.*

Proof. Suppose that

$$c(\rho) = \prod_{v \in V_n \setminus \{0\}} (1 + v)^{\mu(v)}.$$

The non-negative integer $\mu(v)$ is the divisibility of $c(\rho)$ by $1 + v$. In other words, $c(\rho)$ is divisible by $(1 + v)^{\mu(v)}$ but not divisible by $(1 + v)^{\mu(v)+1}$. In order to prove the lemma, it suffices to show that $\mu(v)$ is a constant function of $v \in V_n \setminus \{0\}$. Suppose that $\mu(u) < \mu(v)$ for some $u, v \in V_n \setminus \{0\}$. Let $w \in W$ be an element such that $wv = u$. Then, since w acts trivially on $c(\rho)$, we have

$$c(\rho) = wc(\rho) = \prod_{v' \in V_n \setminus \{0\}} (w(1 + v'))^{\mu(v')} = \left(\prod_{v' \in V_n \setminus \{0, v\}} (1 + wv')^{\mu(v')} \right) (1 + u)^{\mu(v)}.$$

This implies that $\mu(u) \geq \mu(v)$. It is a contradiction. Hence, we have the desired result. \square

By Proposition 2.2 and Lemma 2.3, we have the following result:

Proposition 2.4. *Let G be a compact connected Lie group and let A_n be an elementary abelian p -subgroup of G . Suppose that the Weyl group of A_n , that is the quotient of the normalizer of A_n in G by the centralizer of A_n in G , acts transitively on $V_n \setminus \{0\}$. Then, $B\eta^*(Ch_{H\mathbb{Z}/p}(G)) \subset D_n$, where $\eta : A_n \rightarrow G$ be the inclusion of A_n into G .*

We end this section by recalling the following fact:

Proposition 2.5. *The action of $SL_n(\mathbb{Z}/p)$ on $V_n \setminus \{0\}$ is transitive for $n \geq 2$.*

Proof. It is an easy exercise of linear algebra. It suffices to show that for any $a = (a_1, a_2, \dots, a_n) \in V_n \setminus \{0\}$, there exists a matrix g in $SL_n(\mathbb{Z}/p)$ such that

$$g \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = {}^t a,$$

where ${}^t a$ is the transpose of a . If necessary, applying a permutation, without loss of generality, we may assume that $a_1 \neq 0$. We choose the first column vector of g to be ${}^t a$ and the first row vector of g to be $(a_1, 0, \dots, 0)$ and we choose the rest of the entries in the matrix g so that the matrix obtained from g by removing the first column and the first row is a diagonal matrix whose (i, i) -entry is 1 for $i = 1, \dots, n-2$ and $(n-1, n-1)$ -entry is a_1^{-1} . Then by computing the cofactor expansion along the first row, we see that the determinant of g is 1 and so g is in $SL_n(\mathbb{Z}/p)$. By definition, it is clear that g satisfies the required equality. \square

Thus, in order to prove Theorem 1.1, it suffices to show that there exists an elementary abelian p -subgroup A_n whose Weyl group is $SL_n(\mathbb{Z}/p)$ and that $B\eta^*(x) \notin D_n$. This is what we do in the next section.

3. CHERN SUBRINGS

In this section, we prove Theorem 1.1 by observing the cohomology of non-toral elementary abelian p -subgroup of G . There exist non-toral elementary abelian p -subgroups in a compact connected Lie group if the integral homology of the Lie group has p -torsion. These non-toral elementary abelian p -subgroups and their Weyl groups are known for $(p, G) = (p, PU(n)), (3, F_4), (3, E_6), (3, E_7), (3, E_8), (5, E_8)$. We refer the reader to Andersen et al. [A-G-M-V] and its references. In this paper, we use the following results for $(p, G) = (p, PU(p)), (3, F_4)$ and $(5, E_8)$ only:

Proposition 3.1. *There hold the following:*

- (1) *For $(p, G) = (p, PU(p))$, there exists a non-toral elementary abelian p -subgroup A_2 of rank 2 such that its Weyl group in G is the special linear group $SL_2(\mathbb{Z}/p)$.*
- (2) *For $(p, G) = (3, F_4), (5, E_8)$, there exists a non-toral elementary p -subgroup A_3 of rank 3 such that its Weyl group in G is the special linear group $SL_3(\mathbb{Z}/p)$.*

Let $\eta : A_n \rightarrow G$ be the inclusion of non-toral elementary abelian p -subgroup in G . In [Ka-Ya], we computed the image of the induced homomorphism

$$B\eta^* : H^*(BG; \mathbb{Z}/p) \rightarrow SM_n$$

for $(p, G) = (p, PU(p))$, $n = 2$ and for $(p, G) = (3, F_4), (3, E_6), (3, E_7), (5, E_8)$, $n = 3$. Since we wish to include the case $(p, G) = (3, E_8)$ in Theorem 1.1, instead of making use of the computation of the image of $B\eta^*$, we use the following result, which is also used in the computation of the image of $B\eta^*$:

Proposition 3.2. *There hold the following:*

- (1) *The induced homomorphism*

$$H^2(BPU(p); \mathbb{Z}/p) \rightarrow SM_2^2 = \mathbb{Z}/p\{dt_1 dt_2\}$$

is an isomorphism.

- (2) *For $(p, G) = (3, F_4)$ and $(5, E_8)$, the induced homomorphism*

$$H^4(BG; \mathbb{Z}/p) \rightarrow SM_3^4 = \mathbb{Z}/p\{Q_0(dt_1 dt_2 dt_3)\}$$

is an isomorphism.

Now, we prove Theorem 1.1 for $(p, G) = (3, E_8)$. As we mentioned at the end of the previous section, it suffices to show that $B\eta^*(x) \notin D_3$. There is a sequence of inclusions

$$F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8$$

and the induced homomorphisms

$$H^4(BF_4; \mathbb{Z}/p) \leftarrow H^4(BE_6; \mathbb{Z}/p) \leftarrow H^4(BE_7; \mathbb{Z}/p) \leftarrow H^4(BE_8; \mathbb{Z}/p) = \mathbb{Z}/p$$

are isomorphisms. Recall that we denote the generator of $H^4(BE_8; \mathbb{Z}/3)$ by x_4 . We define $x \in H^{26}(BE_8; \mathbb{Z}/3)$ by $x = Q_1 Q_2(x_4)$. Since the induced homomorphism maps x_4 to $Q_0(dt_1 dt_2 dt_3)$ by Proposition 3.2, it maps x to $e_3 = Q_0 Q_1 Q_2(dt_1 dt_2 dt_3)$ in SD_3 . It is clear that e_3^a is not in D_3 unless a is divisible by $p-1$. Thus, we have Theorem 1.1 for $(p, G) = (3, E_8)$. Theorem 1.1 for the other (p, G) 's can be proved in the same manner.

4. PROOF OF PROPOSITION 1.2

In this section, we prove Proposition 1.2 by computing the second Chern class of the adjoint representation of the exceptional Lie group $\alpha : E_8 \rightarrow SO(248)$. Similar computation was done in [Sc-Ya] for the irreducible representation $F_4 \rightarrow SO(26)$.

Since the induced homomorphism

$$H^4(BF_4; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_6; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_7; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_8; \mathbb{Z}_{(3)}) = \mathbb{Z}_{(3)}$$

are isomorphisms, if $3x_4$ in $H^4(BE_8; \mathbb{Z}_{(3)})$ is a Chern class, so is in $H^4(BG; \mathbb{Z}_{(3)})$ for $G = F_4, E_6, E_7$. So, it suffices to show the proposition for $G = E_8$.

Let $\alpha : E_8 \rightarrow SO(248)$ be the adjoint representation of E_8 . By the construction of the exceptional Lie group E_8 in [Ad], there exists a homomorphism $\beta : \text{Spin}(16) \rightarrow E_8$ such that the induced representation $\alpha \circ \beta$ is the direct sum of $\lambda_{16}^2 : \text{Spin}(16) \rightarrow SO(120)$ and $\Delta_{16}^+ : \text{Spin}(16) \rightarrow SO(128)$. See [Ad, Corollary 7.3] and [Mi-Ni, p. 143]. Let T^8 be the maximal torus of $\text{Spin}(16)$. Let T^1 be the first factor of T^8 and $\eta : T^1 \rightarrow \text{Spin}(16)$ the inclusion of T^1 into $\text{Spin}(16)$. Denote by $R(G)$ the complex representation ring of G . The complexification of λ_{16}^2 corresponds to the second elementary symmetric function of $z_1^2 + z_1^{-2}, \dots, z_8^2 + z_8^{-2}$ in $R(T^8)$ and the complexification of Δ_{16}^+ corresponds to $\sum_{\varepsilon_1 \cdots \varepsilon_8 = 1} z_1^{\varepsilon_1} \cdots z_8^{\varepsilon_8}$ in $R(T^8)$,

where $\varepsilon_r = \pm 1$ for $r = 1, \dots, 8$.

So, the restriction of the complexification of λ_{16}^2 to T^1 corresponds to

$$2^2 \binom{7}{2} + 2 \binom{7}{1} (z_1^2 + z_1^{-2}) = 84 + 14(z_1^2 + z_1^{-2}) \quad \text{in } R(T^1).$$

The restriction of the complexification of Δ_{16}^+ to T^1 corresponds to

$$2^6(z_1 + z_1^{-1}) = 64(z_1 + z_1^{-1}) \quad \text{in } R(T^1).$$

Therefore, the total Chern class of the complexification of $\alpha \circ \beta \circ \eta$ is

$$\{(1+2u)(1-2u)\}^{14} \{(1+u)(1-u)\}^{64} = 1 - 120u^2 + \cdots \in \mathbb{Z}[u] = H^*(BT^1; \mathbb{Z}),$$

where u is the generator of $H^2(BT^1; \mathbb{Z}) = \mathbb{Z}$. Since $120 = 2^3 \cdot 3 \cdot 5$, the Chern class $c_2(\alpha)$ represents $\gamma p x_4$ for $p = 3, 5$ in $H^4(BE_8; \mathbb{Z}_{(p)})$, where γ is a unit in $\mathbb{Z}_{(p)}$ and x_4 is the generator of $H^4(BE_8; \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$. This completes the proof of Proposition 1.2.

REFERENCES

- [Ad] J. F. Adams, *Lectures on exceptional Lie groups*, Univ. Chicago Press, Chicago, IL, 1996.
- [A-G-M-V] K. K. S. Andersen et al., The classification of p -compact groups for p odd, *Ann. of Math.* (2) **167** (2008), no. 1, 95–210.
- [Ka-Mi] M. Kameko and M. Mimura. M \ddot{u} i invariants and Milnor operations, *Geometry and Topology Monographs* **11**, (2007), 107-140.
- [Ka-Ya] M. Kameko and N. Yagita, The Brown-Peterson cohomology of the classifying spaces of the projective unitary groups $PU(p)$ and exceptional Lie groups, *Trans. Amer. Math. Soc.* **360** (2008), no. 5, 2265–2284.
- [Ko-Ya] A. Kono and N. Yagita, Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups, *Trans. Amer. Math. Soc.* **339** (1993), no. 2, 781–798.
- [Mi-Ni] M. Mimura and T. Nishimoto. On the Stiefel-Whitney classes of the representations associated with $Spin(15)$, *Geometry and Topology Monographs* **11**, (2007), 141-176.
- [Sc-Ya] B. Schuster and N. Yagita, Transfers of Chern classes in BP-cohomology and Chow rings, *Trans. Amer. Math. Soc.* **353** (2001), no. 3, 1039–1054 (electronic).
- [Ta] E. Targa, Chern classes are not enough. Appendix to: “On the cohomology and the Chow ring of the classifying space of PGL_p ” by A. Vistoli, *J. Reine Angew. Math.* **610** (2007), 229–233.
- [To] B. Totaro, The Chow ring of a classifying space, in *Algebraic K-theory (Seattle, WA, 1997)*, 249–281, *Proc. Sympos. Pure Math.*, 67, Amer. Math. Soc., Providence, RI.
- [Vi] A. Vistoli, On the cohomology and the Chow ring of the classifying space of PGL_p , *J. Reine Angew. Math.* **610** (2007), 181–227.
- [Ya] N. Yagita, Applications of Atiyah-Hirzebruch spectral sequences for motivic cobordism, *Proc. London Math. Soc.* (3) **90** (2005), no. 3, 783–816.

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